arXiv:1507.08890v1 [math.AP] 31 Jul 2015

THE GENERAL J-FLOWS

WEI SUN

Abstract. We study the general J-flows. We use Moser iteration to obtain the uniform estimate.

1. Introduction

In [5], Donaldson first defined the J-flow in the setting of moment maps. Later in the study of Mabuchi energy, Chen [2] independently defined the J-flow as the gradient flow for the J-functional under the normalization of the I-functional.

Let (M, ω) be a closed Kähler manifold of complex dimension $n \geq 2$, and χ a smooth closed real $(1,1)$ form in Γ^k_ω. Throughout this paper, Γ^k_ω is the set of all the real $(1,1)$ forms whose eigenvalue set with respect to ω belong to k-positive cone in $\mathbb{R}^n$. We consider the general J-flow for $n \geq k > l \geq 1$,

$$\frac{\partial u}{\partial t} = c - \frac{\chi^l_u \wedge \omega^{n-l}}{\chi^k_u \wedge \omega^{n-k}}, \tag{1.1}$$

where

$$c = \frac{\int_M \chi^l \wedge \omega^{n-l}}{\int_M \chi^k \wedge \omega^{n-k}} \qquad \text{and} \qquad \chi_u = \chi + \sqrt{-1}\partial\bar{\partial} u. \tag{1.2}$$

Indeed, the results of this paper apply to more general forms

$$\frac{\partial u}{\partial t} = c - \frac{\sum_{l=0}^{k-1} b_l \chi^l_u \wedge \omega^{n-l}}{\chi^k_u \wedge \omega^{n-k}}, \quad b_l \geq 0 \text{ and } \sum_{l=0}^{k-1} b_l > 0. \tag{1.3}$$

We recall the general J-functionals, which was actually defined by Fang, Lai and Ma [6]. Let $\mathcal{H}$ be the space

$$\mathcal{H} := \{u \in C^\infty(M) \mid \chi_u \in \Gamma^k_\omega\}. \tag{1.4}$$

For any curve $v(s) \in \mathcal{H}$, we define the funtional J_m by

$$\frac{dJ_m}{ds} = \int_M \frac{\partial v}{\partial s} \chi^m_v \wedge \omega^{n-m} \tag{1.5}$$

for any $0 \leq m \leq k$. The parabolic flow (1.1) can thus be viewed as the negative gradient flow for the J_l-functional under the normalization $J_k(u) = 0$.

Project Funded by China Postdoctoral Science Foundation.

It is easy to see that the J-flow is a special case,

$$\frac{\partial u}{\partial t} = c - \frac{\chi_u^{n-1} \wedge \omega}{\chi_u^n}. \tag{1.6}$$

Chen [3] proved the long time existence of the solution to the J-flow. Weinkove [13] [14] showed the convergence under a strong condition. In [8], Song and Weinkove put forward and established a necessary and sufficient condition for convergence, which is called the cone condition. A numerical version of the cone condition, which is easier to check in concrete examples, was proposed by Lejmi and Székelyhidi [7]. Later, Collins and Székelyhidi [4] affirmed the numerical cone condition on toric manifolds.

In this paper, we prove the following theorem.

Theorem 1.1. *Let (M, ω) be a closed Kähler manifolds of complex dimension $n \geq 2$ and χ a closed form in Γ_ω^k. Suppose that there exists $\chi' \in [\chi] \cap \Gamma_\omega^k$ satisfying the cone condition*

$$ck\chi'^{k-1} \wedge \omega^{n-k} > l\chi'^{l-1} \wedge \omega^{n-l}. \tag{1.7}$$

Suppose that u is the solution to the general J-flow (1.1) on maximal time $[0, T)$. Then here exists a uniform constant $C > 0$ such that for any $t \in [0, T)$

$$\sup_M u(x,t) - \inf_M u(x,t) < C. \tag{1.8}$$

We shall apply the Moser iteration approach in [10] and an idea from Blocki [1] and Székelyhidi [12]. In order to obtain a time-independent uniform estimate, we apply the Moser iteration locally in time t. Since all other arguments are the same as those in [11], we immediately have a corollary. In [11], we applied the ABP estimate to obtain the uniform estimate.

Corollary 1.2. *Under the assumption of Theorem 1.1, there exists a long time solution u to the general J-flow (1.1). Moreover, the solution u converges in C^∞ to u_∞ with $\chi_{u_\infty} \in \Gamma_\omega^k$ satisfying*

$$\chi_{u_\infty}^l \wedge \omega^{n-l} = c\chi_{u_\infty}^k \wedge \omega^{n-k}. \tag{1.9}$$

2. The uniform estimate

As in [8] [6], the cone condition (1.7) is necessary and sufficient. In other words, the cone condition means that there is a C^2 function v such that $\chi_v \in \Gamma_\omega^k$ and

$$ck\chi_v^{k-1} \wedge \omega^{n-k} > l\chi_v^{l-1} \wedge \omega^{n-l}. \tag{2.1}$$

We may assume that there is $c > \epsilon > 0$ such that

$$(c - 2\epsilon)k\chi_v^{k-1} \wedge \omega^{n-k} > l\chi_v^{l-1} \wedge \omega^{n-l}. \tag{2.2}$$

Without loss of generality, we may also assume that $\sup_M v = -2\epsilon$.

Along the solution flow $u(x,t)$ to equation (1.1),

$$\begin{aligned}\frac{d}{dt}J_k(u) &= \int_M \frac{\partial u}{\partial t}\chi_u^k \wedge \omega^{n-l} \\ &= c\int_M \chi_u^k \wedge \omega^{n-k} - \int_M \chi_u^l \wedge \omega^{n-l} = 0.\end{aligned} \tag{2.3}$$

Lemma 2.1. *At any time t,*

$$0 \le \sup_M u \le -C_1 \inf_M u + C_2 \quad \text{and} \quad \inf_M u(x,t) \le 0. \tag{2.4}$$

Proof. Choosing the path $v(s) = su$ and noting that $J_k(u) = 0$,

$$\frac{1}{k+1}\sum_{i=0}^{k}\int_M u\chi_u^i \wedge \chi^{k-i} \wedge \omega^{n-k} = 0. \tag{2.5}$$

The first and third inequalities in (2.4) then follow from (2.5) and Gårding's inequality.

Rewriting (2.5),

$$\int_M u\chi^k \wedge \omega^{n-k} = -\sum_{i=1}^{k}\int_M u\chi_u^i \wedge \chi^{k-i} \wedge \omega^{n-k}. \tag{2.6}$$

Let C be a positive constant such that

$$\omega^n \le C\chi^k \wedge \omega^{n-l}. \tag{2.7}$$

Then as in [9],

$$\begin{aligned}\int_M u\omega^n &= \int_M \left(u - \inf_M u\right)\omega^n + \int_M \inf_M u\omega^n \\ &\le C\int_M \left(u - \inf_M u\right)\chi^k \wedge \omega^{n-k} + \inf_M u \int_M \omega^n \\ &\le \inf_M u\left(\int_M \omega^n - (k+1)C\int_M \chi^{n-\alpha}\wedge\omega^\alpha\right).\end{aligned} \tag{2.8}$$

The second inequality in (2.4) then follows from (2.8) (see Yau [15]).

□

Proof of Theorem 1.1. According to Lemma 2.1, it suffices to prove a lower bound for $\inf_M(u-v)(x,t)$. We claim that

$$\inf_M (u-v)(x,t) > \inf_M u_t(x,0) - C_0, \tag{2.9}$$

where C_0 is to be specified later.

Differentiating the general J-flow (1.1) with respect to t,

$$\frac{\partial u_t}{\partial t} = \frac{\chi_u^l \wedge \omega^{n-l}}{\chi_u^k \wedge \omega^{n-k}} \left(\frac{k\sqrt{-1}\partial\bar{\partial} u_t \wedge \chi_u^{k-1} \wedge \omega^{n-k}}{\chi_u^k \wedge \omega^{n-k}} - \frac{l\sqrt{-1}\partial\bar{\partial} u_t \wedge \chi_u^{l-1} \wedge \omega^{n-l}}{\chi_u^l \wedge \omega^{n-l}} \right), \tag{2.10}$$

which is also parabolic. Applying the maximum principle, u_t reaches the extremal values at $t = 0$. So when $t \leq 1$, we have

$$\inf_M (u - v)(x, t) \geq \inf_M u(x, t) + 2\epsilon \geq t \inf_M u_t(x, 0) + 2\epsilon. \tag{2.11}$$

Therefore if (2.9) fails, there must be time $t_0 > 1$ such that

$$\inf_M (u - v)(x, t_0) = \inf_{M\times[0,t_0]} (u - v)(x, t) = \inf_M u_t(x, 0) - C_0 \leq 0. \tag{2.12}$$

For $p \geq 1$, we consider the integral

$$I = \int_M \varphi^p \Big[\Big((c - u_t)\chi_u^k \wedge \omega^{n-k} - (c - \epsilon)\chi_v^k \wedge \omega^{n-k} \Big) - \Big(\chi_u^l \wedge \omega^{n-l} - \chi_v^l \wedge \omega^{n-l} \Big) \Big]. \tag{2.13}$$

It is easy to see that form some constant $C > 0$

$$I \leq C \int_M \varphi^p \omega^n. \tag{2.14}$$

$$I = \int_M \varphi^p \Big[(c - \epsilon - su_t + s\epsilon)\chi_{su+(1-s)v}^k \wedge \omega^{n-k} - \chi_{su+(1-s)v}^l \wedge \omega^{n-l} \Big] \Bigg|_{s=0}^1 \tag{2.15}$$

For simplicity, we denote $\chi_s = \chi_{su+(1-s)v}$. Thus

$$\begin{aligned} I = & \int_0^1 ds \int_M \varphi^p \sqrt{-1}\partial\bar{\partial}(u - v) \wedge \Big[k(c - \epsilon - su_t + s\epsilon)\chi_s^{k-1} \wedge \omega^{n-k} - l\chi_s^{l-1} \wedge \omega^{n-l} \Big] \\ & - \int_0^1 ds \int_M \varphi^p (u_t - \epsilon)\chi_s^k \wedge \omega^{n-k} \\ = & -p \int_0^1 ds \int_M \varphi^{p-1} \sqrt{-1}\partial\varphi \wedge \bar{\partial}(u - v) \\ & \qquad \wedge \Big[k(c - \epsilon - su_t + s\epsilon)\chi_s^{k-1} \wedge \omega^{n-k} - l\chi_s^{l-1} \wedge \omega^{n-l} \Big] \\ & - \int_0^1 ds \int_M ks\varphi^p \sqrt{-1}\bar{\partial}(u - v) \wedge \partial u_t \wedge \chi_s^{k-1} \wedge \omega^{n-k} \\ & - \int_0^1 ds \int_M \varphi^p (u_t - \epsilon)\chi_s^k \wedge \omega^{n-k}. \end{aligned} \tag{2.16}$$

We define

$$\varphi = (u - v - \epsilon t - L)^- \geq 0, \tag{2.17}$$

where L to be specified later, and hence

$$\begin{aligned} I = p\int_0^1 ds \int_M & \varphi^{p-1}\sqrt{-1}\partial(u-v)\wedge\bar\partial(u-v) \\ & \wedge \Big[k(c-\epsilon-su_t+s\epsilon)\chi_s^{k-1}\wedge\omega^{n-k} - l\chi_s^{l-1}\wedge\omega^{n-l}\Big] \\ & - \int_0^1 ds\int_M ks\varphi^p\sqrt{-1}\bar\partial(u-v)\wedge\partial u_t\wedge\chi_s^{k-1}\wedge\omega^{n-k} \\ & + \frac{1}{p+1}\int_0^1 ds\int_M \partial_t(\varphi^{p+1})\chi_s^k\wedge\omega^{n-k}. \end{aligned} \tag{2.18}$$

Using integration by parts again, we know that almost everywhere over time t,

$$\begin{aligned} & -\int_0^1 ds\int_M ks\varphi^p\sqrt{-1}\bar\partial(u-v)\wedge\partial u_t\wedge\chi_s^{k-1}\wedge\omega^{n-k} \\ & = \frac{1}{p+1}\int_0^1 ds\int_M ks\varphi^{p+1}\sqrt{-1}\partial\bar\partial u_t\wedge\chi_s^{k-1}\wedge\omega^{n-k} \\ & = \frac{1}{p+1}\int_0^1 ds\int_M \varphi^{p+1}\partial_t(\chi_s^k\wedge\omega^{n-k}), \end{aligned} \tag{2.19}$$

and thus

$$\begin{aligned} I = p\int_0^1 ds \int_M & \varphi^{p-1}\sqrt{-1}\partial(u-v)\wedge\bar\partial(u-v) \\ & \wedge \Big[k(c-\epsilon-su_t+s\epsilon)\chi_s^{k-1}\wedge\omega^{n-k} - l\chi_s^{l-1}\wedge\omega^{n-l}\Big] \\ & + \frac{1}{p+1}\frac{d}{dt}\int_0^1 ds\int_M \varphi^{p+1}\chi_s^k\wedge\omega^{n-k}. \end{aligned} \tag{2.20}$$

Since $-S_{l-1;i}/S_{k-1;i}$ and $S_{k-1;i}^{\frac{1}{k-1}}$ are concave, we have

$$\begin{aligned} k(c-\epsilon-su_t+s\epsilon)\chi_s^{k-1}\wedge\omega^{n-k} - l\chi_s^{l-1}\wedge\omega^{n-l} & \geq (1-s)\epsilon k\chi_s^{k-1}\wedge\omega^{n-k} \\ & \geq (1-s)^k\epsilon k\chi_v^{k-1}\wedge\omega^{n-k}. \end{aligned} \tag{2.21}$$

Then

$$\begin{aligned} I \geq \frac{\epsilon kp}{k+1}\int_M & \varphi^{p-1}\sqrt{-1}\partial(u-v)\wedge\bar\partial(u-v)\wedge\chi_v^{k-1}\wedge\omega^{n-k} \\ & + \frac{1}{p+1}\frac{d}{dt}\int_0^1 ds\int_M \varphi^{p+1}\chi_s^k\wedge\omega^{n-k}. \end{aligned} \tag{2.22}$$

Integrating I from t_0-1 to $t'\in[t_0-1,t_0]$, we obtain

$$\begin{aligned} C\int_{t_0-1}^{t'} dt\int_M \varphi^p\omega^n \geq \sigma p & \int_{t_0-1}^{t'} dt\int_M \varphi^{p-1}\sqrt{-1}\partial(u-v)\wedge\bar\partial(u-v)\wedge\omega^{n-1} \\ & + \frac{1}{p+1}\int_0^1 ds\int_M \varphi^{p+1}\chi_s^k\wedge\omega^{n-k}\Big|_{t=t_0-1}^{t'}. \end{aligned} \tag{2.23}$$

Choosing $L = \inf_{M\times[0,t_0]}(u-v) - \epsilon t_0 + \epsilon$,
$$\varphi(x, t_0 - 1) = (u - v - \inf_{M\times[0,t_0]}(u-v))^- = 0, \tag{2.24}$$
and
$$\sup_{M\times[0,t_0]} \varphi(x,t) = \sup_M \varphi(x,t_0) = (u(x_i,t_0) - v(x_i) - \epsilon - \inf_{M\times[0,t_0]}(u-v))^- = \epsilon. \tag{2.25}$$
So
$$\begin{aligned} C\int_{t_0-1}^{t'} dt \int_M \varphi^p \omega^n \geq{}& \sigma p \int_{t_0-1}^{t'} dt \int_M \varphi^{p-1}\sqrt{-1}\partial(u-v)\wedge\bar\partial(u-v)\wedge\omega^{n-1} \\ &+ \frac{1}{p+1}\int_0^1 ds \int_M \varphi^{p+1}\chi_s^k \wedge \omega^{n-k}\Big|_{t=t'}. \end{aligned} \tag{2.26}$$
By integration by parts, we observe that
$$\int_0^1 ds \int_M \varphi^{p+1}\chi_s^k\wedge\omega^{n-k} \geq \int_0^1 ds \int_M \varphi^{p+1}\chi_v^k\wedge\omega^{n-k} = \int_M \varphi^{p+1}\chi_v^k\wedge\omega^{n-k}. \tag{2.27}$$
Substituting (2.26) into (2.27)
$$\begin{aligned} C\int_{t_0-1}^{t'} dt \int_M \varphi^p \omega^n \geq{}& \sigma p \int_{t_0-1}^{t'} dt \int_M \varphi^{p-1}\sqrt{-1}\partial(u-v)\wedge\bar\partial(u-v)\wedge\omega^{n-1} \\ &+ \frac{1}{p+1}\int_M \varphi^{p+1}\chi_v^k \wedge \omega^{n-k}\Big|_{t=t'} \\ \geq{}& \frac{2\sigma}{p+1}\int_{t_0-1}^{t'} dt \int_M \sqrt{-1}\partial\varphi^{\frac{p+1}{2}}\wedge\bar\partial\varphi^{\frac{p+1}{2}}\wedge\omega^{n-1} \\ &+ \frac{1}{p+1}\int_M \varphi^{p+1}\chi_v^k \wedge \omega^{n-k}\Big|_{t=t'}. \end{aligned} \tag{2.28}$$
Consequently,
$$\begin{aligned} C(p+1)\int_{t_0-1}^{t_0} dt \int_M \varphi^p \omega^n \geq{}& 2\sigma \int_{t_0-1}^{t_0} dt \int_M \sqrt{-1}\partial\varphi^{\frac{p+1}{2}}\wedge\bar\partial\varphi^{\frac{p+1}{2}}\wedge\omega^{n-1} \\ &+ \sup_{t\in[t_0-1,t_0]}\int_M \varphi^{p+1}\chi_v^k\wedge\omega^{n-k}. \end{aligned} \tag{2.29}$$
By Sobolev inequality, for $\beta = \frac{n+1}{n}$,
$$C(p+1)\int_{t_0-1}^{t_0} dt \int_M \varphi^p\omega^n \geq \left(\int_{t_0-1}^{t_0} dt \int_M \varphi^{(p+1)\beta}\right)^{\frac{1}{\beta}}. \tag{2.30}$$
We can then iterate $\beta \to \beta^2 + \beta \to \beta^3 + \beta^2 + \beta \to \cdots$ and obtain
$$p_m = \frac{\beta(\beta^{m+1}-1)}{\beta - 1} \tag{2.31}$$

and

$$(\ln C - \ln \beta) + \ln p_{m+1} + p_m \ln ||\varphi||_{L^{p_m}} \geq \frac{p_{m+1}}{\beta} \ln ||\varphi||_{L^{p_{m+1}}}. \tag{2.32}$$

From (2.32),

$$\sum_{m=0}^{q} \frac{\ln C - \ln \beta}{\beta^m} + \sum_{m=0}^{q} \frac{\ln p_{m+1}}{\beta^m} + \beta \ln ||\varphi||_{L^\beta} \geq \frac{p_{q+1}}{\beta^{q+1}} \ln ||\varphi||_{L^{p_{q+1}}}, \tag{2.33}$$

that is

$$\ln\Big(\frac{C}{\beta - 1}\Big) \sum_{m=0}^{q} \frac{1}{\beta^m} + \sum_{m=0}^{q} \frac{\ln(\beta^{m+1} - 1)}{\beta^m} + \beta \ln ||\varphi||_{L^\beta} \geq \frac{\beta^{q+2} - 1}{\beta^q(\beta - 1)} \ln ||\varphi||_{L^{p_{q+1}}}. \tag{2.34}$$

Letting $q \to \infty$,

$$C + \ln ||\varphi||_{L^\beta} \geq \frac{\beta}{\beta - 1} \ln ||\varphi||_{L^\infty} = \frac{\beta \ln \epsilon}{\beta - 1}. \tag{2.35}$$

Therefore, there exists a uniform constant $c_1 > 0$ such that

$$\int_{t_0-1}^{t_0} dt \int_M \varphi^\beta \omega^n \geq c_1. \tag{2.36}$$

So we have

$$\epsilon^\beta \int_{t_0-1}^{t_0} dt \int_{\{\varphi > 0\}} \omega^n \geq c_1. \tag{2.37}$$

When $\varphi > 0$,

$$u < v + \epsilon t + \inf_{M \times [0,t_0]} (u - v) - \epsilon t_0 + \epsilon \leq v + \epsilon + \inf_{M \times [0,t_0]} (u - v) < \inf_{M \times [0,t_0]} (u - v). \tag{2.38}$$

Thus, using an idea from Blocki [1] and Székelyhidi [12],

$$\begin{aligned} c_1 &\leq \epsilon^\beta \int_{t_0-1}^{t_0} \frac{||u^-(x,t)||_{L^1}}{|\inf_{M \times [0,t_0]}(u - v)|} dt \\ &\leq \epsilon^\beta \int_{t_0-1}^{t_0} \frac{||u(x,t) - \sup_M u(x,t)||_{L^1}}{|\inf_{M \times [0,t_0]}(u - v)|} dt. \end{aligned} \tag{2.39}$$

Since Δu has a lower bound, we have a uniform bound for $||u(x,t) - \sup_M u(x,t)||_{L^1}$. As a consequence, there is a uniform constant $C_0 > 0$ such that

$$\inf_{M \times [0,t_0]} (u - v) > -C_0. \tag{2.40}$$

However, it contradicts the definition of t_0.

□

Acknowledgements The author is very grateful to Bo Guan for his advice and encouragement. The author also wishes to thank Hongjie Dong for helpful discussions.

References

[1] Z. Blocki, *On uniform estimate on Calabi-Yau theorem*, Sci. China Ser. A, **48** (2005), 244–247.
[2] X.-X. Chen, *On the lower bound of the Mabuchi energy and its application*, Int. Math. Res. Notices **12** (2000), 607–623
[3] X.-X. Chen, *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom. **12** (2004), 837–852.
[4] T. C. Collins and G. Székelyhidi, *Convergence of the J-flow on toric manifolds*, preprint, arXiv:1412.4809.
[5] S. K. Donaldson, *Moment maps and diffeomorphisms*, Asian J. Math. **3** (1999), 1–16.
[6] H. Fang, M.-J. Lai and X.-N. Ma, *On a class of fully nonlinear flows in Kähler geometry* J. Reine Angew. Math. **653** (2011), 189–220.
[7] M. Lejmi and G. Székelyhidi, *The J-flow and stability*, preprint, arXiv:1309.2821.
[8] J. Song and B. Weinkove, *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math. **61** (2008), 210–229.
[9] W. Sun, *Parabolic complex Monge-Ampère type equations on closed Hermtian manifolds*, preprint, arXiv:1311.3002.
[10] W. Sun, *On a class of fully nonlinear elliptic equations on closed Hermitian manifolds II: L^∞ estimate*, preprint, arXiv:1407.7630.
[11] W. Sun, *The parabolic flows for complex quotient equations*, preprint.
[12] G. Székelyhidi, *Fully non-linear elliptic equations on compact Hermitian manifolds*, preprint, arXiv:1501.02762.
[13] B. Weinkove, *Convergence of the J-flow on Kähler surfaces*, Comm. Anal. Geom. **12** (2004), 949–965.
[14] B. Weinkove, *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Differential Geom. **73** (2006), 351–358.
[15] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.* Comm. Pure Appl. Math. **31** (1978), 339–411.

Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200433, China
E-mail address: sunwei_math@fudan.edu.cn